The complementary cubic

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1 Introduction

It is a well known property of the univariant cubic polynomial that the abscissa of the point of contact T of the tangent from a real root (see Figure 1) yields the real component of the associated pair of complex roots [1,2,3,4]. Indeed, not only does this apply to all polynomials [5], but it is also a particular (one-step) instance of the Newton-Raphson method for locating a cubic root [6].

In this Note we show (a) that in the case of the cubic this tangent property is related to a seemingly overlooked curve which, for want of a better name, we call the 'complementary cubic', and (b) that T has the same abscissa as a real root of the complementary cubic.

2 The complementary cubic

Since the cubic has three roots there are just three different ways of selecting pairs of roots. Consequently, there are three different loci of the midpoint between pairs of roots, with two of these loci moving continuously between the real and complex planes. The remaining locus (the complementary cubic) lies wholly within the real xy-plane, and (a) passes through both turning points and the point of inflection and (b) is also a cubic polynomial (see Figure 1).

Let $F(X) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients ($a \neq 0$) and roots R_k ($k = 1,2,3$). Let $N(X_N, Y_N)$ be the point of inflection, and let δ and h be the distances shown in Figure 1, as in [7]. These are:

$$\begin{cases} X_N = \dfrac{-b}{3a}, \\ Y_N = F(X_N), \\ \delta^2 = \dfrac{(b^2 - 3ac)}{9a^2}, \\ h = 2a\delta^3. \end{cases}$$

Let $C(x)$ (complementary cubic, shown dashed in Fugure 1) be the locus of the midpoint M of pairs of roots R_2, R_3 as the real root R_1 moves on $F(x)$. When the x-axis lies *outside* the two turning points ($Y_N^2 > h^2$) then M is the midpoint of the two complex roots. Conversely, when the x-axis lies either *between* the two turning points or on one of them ($Y_N^2 \leq h^2$) then M is the midpoint of the two remaining real roots.

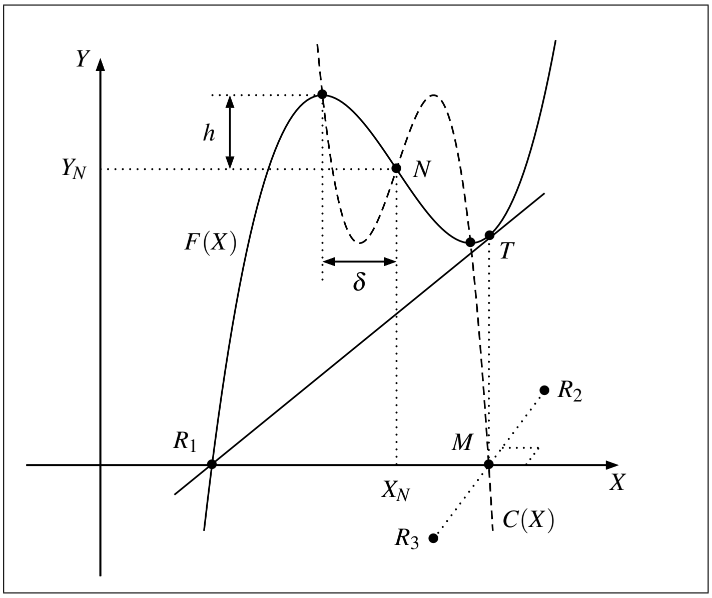

Figure 1:

3 Transformation

The complementary cubic $C(x)$ is generated from the parent cubic $F(x)$ by the linear transformation

$$\begin{cases} x \to 3X_N - 2x, \\ y \to y. \end{cases}$$

The derivation can be regarded as consisting of the following steps:

1. Starting with the cubic $F(x)$, generate the reduced form $f(x)$ using the translation $x \to X_N + x$, giving $f(x) = F(X_N + x)$.

2. Next, generate the complementary reduced form $c(x)$ by halving the abscissa and reflecting the curve about the y-axis through N (point of inflection) using the substitution $x \to -2x$, giving $c(x) = F(X_N - 2x)$.

3. Finally, generate the required complementary form $C(x)$ by sending $c(x)$ to the original axis frame using the reverse translation $x \to x - X_N$, giving $C(x) = F(X_N - 2(x - X_N)) = F(3X_N - 2x)$, and hence $C(x)$ is a cubic polynomial.

For example, the details of the two curves $F(x)$ and $C(x)$ shown in Figure 1 are as follows:

$$\begin{cases} F(x) = x^3 - 12x^2 + 45x - 44, \\ \quad X_N = -b/(3a) = 4, \\ \quad Y_N = F(X_N) = 8, \\ \quad \delta^2 = (b^2 - 3ac)/(9a^2) = 1, \\ \quad\; h = 2a\delta^3 = 2, \\ C(x) = F(3X_N - 2x), \\ \qquad = F(12 - 2x), \\ \qquad = -8x^3 + 96x^2 - 378x + 496. \end{cases}$$

4 Tangent property

The above transformation can be used to show that T has the same abscissa as one of the real roots of the complementary cubic $C(x)$, as follows.

Since the tangent property is independent of the root configuration of $F(x)$, we can, with no loss of generality, use the configuration shown in Figure 1. Let X_N, X_T, X_M denote the x-coordinates of the points N, T, M respectively, and let R_1 denote a real root of $F(x)$. Since $C(x) = F(3X_N - 2x)$ (see above), we can write

$$\begin{aligned} C(X_T) &= F(3X_N - 2X_T), \\ &= F(3X_N - 2X_M), \end{aligned}$$

since it is standard that $X_T = X_M$ [1,2,3,4]. We also note in passing that the above expression is essentially equivalent to [3, equation 2].

By setting $X_M = X_N + s$ we obtain

$$\begin{aligned} C(X_T) &= F(3X_N - 2(X_N + s)), \\ &= F(X_N - 2s). \end{aligned}$$

But, $X_N - 2s = R_1$; and since $F(R_1) = 0$ then $C(X_T) = 0$. Since $X_T = X_M$, it follows that M coincides with a root of $C(x)$ (the real zero as shown in Figure 1). Consequently, T has the same abscissa as one of the real zeros of $C(x)$.

5 Example

Starting with $F(x) = x^3 - 12x^2 + 45x - 44$ (where $X_N = 4$) as shown in Figure 1, we can illustrate the above tangent property by generating the simpler reduced forms $f(x)$ and $c(x)$ (for which $X_N = 0$) and then determining their zeros, as follows.

$$\begin{cases} f(x) = F(x + X_N), \\ \quad\;\; = F(x+4), \\ \quad\;\; = x^3 - 3x + 8, \quad \text{(zeros: } \approx -2{\cdot}492,\ 1{\cdot}246 \pm 1{\cdot}287i), \\ c(x) = f(3X_N - 2x), \\ \quad\;\; = f(-2x), \\ \quad\;\; = -8x^3 + 6x + 8, \quad \text{(zeros: } \approx -0{\cdot}623 \pm 0{\cdot}643i,\ 1{\cdot}246). \end{cases}$$

Comparing the zeros of $f(x)$ and $c(x)$ shows that the abscissa of the midpoint of the complex zeros of $f(x)$ ($\approx 1{\cdot}246$) is equal to that of the real zero of its complementary cubic $c(x)$.

6 Conclusion

Although similar real xy-plane midpoint loci which pass through the turning points can be constructed for other polynomials, that for the cubic is interesting in that not only is it a polynomial, but it also has the same degree as the parent curve. It also serves as a useful guide for the visualisation of complex zeros.

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7 References

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