Feedback—On Note 92.35

Real roots of cubics: explicit formula for quasi-solutions

RWD Nickalls [1]

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http://www.nickalls.org/dick/papers/maths/cubictables2009.pdf

RWD Nickalls writes: I was very interested in the way Edgar Rechtschaffen [2] tabulated the roots of the cubic $w^3 - 3w + 2\alpha = 0$ in terms of the parameter α, since until the advent of personal computers, cubics were commonly solved using one of the many available tables [1] developed for solving equations numerically.

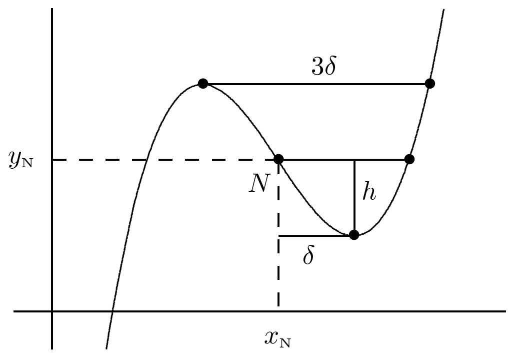

Figure 1:

In fact the reason such a tabulation can be based on Rechtschaffen's parameter α lies in a generally overlooked geometric feature of the cubic $f(x) \equiv ax^3 + bx^2 + cx + d$, namely that $h = 2a\delta^3$ (see figure 1 [3]). This relationship immediately suggests an elegant solution [2] to the (now historical) problem of tabulating the roots of the reduced cubic in a simple generic form (see Table 1), since dividing the cubic's reduced geometric form [3] $az^3 - 3a\delta^2 z + y_N = 0$ throughout by h (excluding the trivial $h = \delta = 0$ case), yields the useful normalised form

$$\left(\frac{z}{\delta}\right)^3 - 3\left(\frac{z}{\delta}\right) + 2\frac{y_N}{h} = 0. \tag{1}$$

Thus α is revealed to be the ratio y_N/h, and we can begin to see why this method works.

[1]Department of Anaesthesia, Nottingham University Hospitals, City Hospital Campus, Nottingham, UK. email: dick@nickalls.org

[2] E. Rechtschaffen (2008). NOTE 92.35 Real roots of cubics: explicit formula for quasi-solutions. *The Mathematical Gazette*; **92** (July 2008), pp. 268–276 (JSTOR).

If $f(x) = 0$ is the cubic to be solved then the roots z_i $(i = 1, 2, 3)$ of its reduced form can be expressed as multiples of δ (equivalent to what Rechtschaffen terms x_m), i.e., $x_i = x_N + z_i = x_N + \delta(z_i/\delta)$ where [3] $x_N = -b/(3a)$ and $\delta^2 = (b^2 - 3ac)/(9a^2)$. Thus Rechtschaffen's tabulated values of $r_i(\alpha)$ for values of α are exactly equivalent to the more intuitive z_i/δ associated with y_N/h, as shown in Table 1 below.

y_N/h	z_1/δ	z_2/δ	z_3/δ
±2·00	∓2·195823345446	±1·097911672723 ± j 0·785003263244	
±1·75	∓2·151069024856	±1·075534512428 ± j 0·685801328573	
±1·50	∓2·103803402736	±1·051901701368 ± j 0·565235851677	
±1·25	∓2·053621575879	±1·026810779106 ± j 0·403758818630	
±1·00	∓2	±1	±1
±0·75	∓1·942241850970	±0·557874698332	±1·384367152638
±0·50	∓1·879385241572	±0·347296355334	±1·532088886238
±0·25	∓1·810037929234	±0·168254401781	±1·641783527453
0	$-\sqrt{3}$	0	$\sqrt{3}$

Table 1: (modified from [2])

Finally, Rechtschaffen's lemma 1 stems from the fact that any one of the roots z_i, say z_1, can be represented by $z_1 = p + p'$ where $pp' = \delta^2$, and hence the remaining two roots are $p\omega + p'\omega^2$, $p\omega^2 + p'\omega$, where ω is a complex cube root of unity, as follows.

$$
\begin{aligned}
z_2, z_3 &= p\left(-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}\right) + p'\left(-\frac{1}{2} \mp j\frac{\sqrt{3}}{2}\right), \\
&= -\frac{p+p'}{2} \pm j\frac{\sqrt{3}}{2}\sqrt{(p+p')^2 - 4pp'}, \\
&= -\frac{z_1}{2} \pm \frac{\sqrt{3}}{2}\sqrt{4\delta^2 - {z_1}^2}.
\end{aligned}
$$

Thus given *any* root of (1), say z_1/δ, then the remaining two roots are given by

$$
\frac{z_2}{\delta},\ \frac{z_3}{\delta} = -\frac{z_1}{2\delta} \pm \frac{1}{2}\sqrt{12 - 3\left(\frac{z_1}{\delta}\right)^2}.
$$

Example

Solve $f(x) \equiv x^3 - 9x^2 + 15x + 17 = 0$

The key parameters are: $a = 1$, $x_N = 3$, $y_N = f(x_N) = 8$, $\delta = 2$, $h = 2a\delta^3 = 16$. Thus $y_N/h = 0{\cdot}5$ and so using Table 1 the roots $x_i = x_N + \delta(z_i/\delta)$ are given by

$$\begin{aligned} x_1 &= 3 + 2(-1{\cdot}879385241572) = -0{\cdot}7587, \\ x_2 &= 3 + 2(0{\cdot}347296355334) \quad = \quad 3{\cdot}6945, \\ x_3 &= 3 + 2(1{\cdot}532088886238) \quad = \quad 6{\cdot}0641. \end{aligned}$$

References

1. Salzer H.E., Richards C.H. and Arsham I. *Table for the solution of cubic equations.* pp. 161. (McGraw-Hill Book Company Inc., New York, London, 1958)

2. R. W. D. Nickalls. Solving the cubic using tables. *Theta*; **10** (No. 2, Autumn, 1996) pp. 21–24. (Pub: Mathematics Department, Manchester Metropolitan University, UK) [ISSN: 0953–0738] http://www.nickalls.org/dick/papers/maths/cubictables1996.pdf

3. R. W. D. Nickalls. A new approach to solving the cubic: Cardan's solution revealed. *Math. Gaz.*, **77** (November 1993) pp. 354–359 (JSTOR). http://www.nickalls.org/dick/papers/maths/cubic1993.pdf http://www.jstor.org/stable/3619777

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