The quadratic formula is a particular case

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1 Introduction

When it comes to solving quadratic equations the familiar *quadratic formula* (1) is certainly a faithful friend [1].

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1}$$

However, I would like to suggest that there is more to our friendly formula than meets the eye, and that it can be viewed with advantage as being a *particular case* of a more general formula (2) which gives the location of pairs of certain anatomical parts of polynomials. I discovered this interesting connection while investigating geometric discriminants [3], and have since found the more general formula a useful tool for revealing the basic anatomy of polynomials.

2 General case

Let $f(x)$ be a general polynomial of degree n, and let us differentiate it twice as follows.

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots$$

$$f''(x) = n(n-1)ax^{n-2} + (n-1)(n-2)bx^{n-3} + (n-2)(n-3)cx^{n-4} + \cdots$$

Suppose, for example, that the polynomial is a quartic. In this case $n = 4$ and $f''(x)$ will therefore be a quadratic whose roots will be the two points of inflection of the quartic.

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Solving this in the usual way gives

$$\frac{-b(n-1)(n-2) \pm \sqrt{b^2(n-1)^2(n-2)^2 - 4acn(n-1)(n-2)(n-3)}}{2n(n-1)a},$$

$$= \frac{-b\left(\frac{n-2}{2}\right) \pm \sqrt{b^2\left(\frac{n-2}{2}\right)^2 - \left(\frac{2n}{n-1}\right)\left(\frac{(n-2)(n-3)}{2}\right)ac}}{na}.$$

Since we started with a quartic ($n = 4$) and differentiated it successively to a quadratic, then $(n-2) = 2$ and $(n-3) = 1$, and so some further cancelling can be performed to give the following formula (2) for the x-coordinates of the pair of inflection points of the quartic.

$$\frac{-b}{na} \pm \frac{\sqrt{b^2 - \left(\frac{2n}{n-1}\right)ac}}{na} \qquad (2)$$

However, it is easily shown that the derived formula (2) is independent of the degree of the original polynomial ($n \geq 2$). For example if $n = 5$, although the polynomial will have to be differentiated three times in order to reach the quadratic stage, the extra factors in the leading coefficient will cancel leaving the divisor na as before.

Relationship (2) is therefore a general formula yielding two symmetrical objects, namely a pair of roots, turning points, or points of inflection, depending on whether the polynomial is of degree 2, 3 or 4 respectively. The quadratic formula (1) can therefore usefully be regarded as being a particular case of the general formula (2).

In practice, it is useful to emphasise the different roles of the two components of the general formula (2). The first part $(-b/na)$ is simply the general formula for the x-coordinate (x_N) of what I call the N-point of a polynomial—namely the point to which the axis must be moved in order to make the sum of the roots equal to zero (see [2]). Exactly what the N-point represents depends on the degree of the polynomial; for example N represents the root of a linear equation ($n = 1$), the turning point of a quadratic ($n = 2$), the the point of inflection of a cubic ($n = 3$), and so on. The second component of the general formula (2) gives the symmetrical displacement of the other two objects relative to the N-point.

3 The quadratic

Substituting $n = 2$ in the general formula (2) generates the familiar quadratic formula (1), which of course gives the x-coordinates of the pair of roots x_1, x_2, and indicates that $b^2 - 4ac$ is the algebraic discriminant ∂_2 *of the roots of the quadratic*.

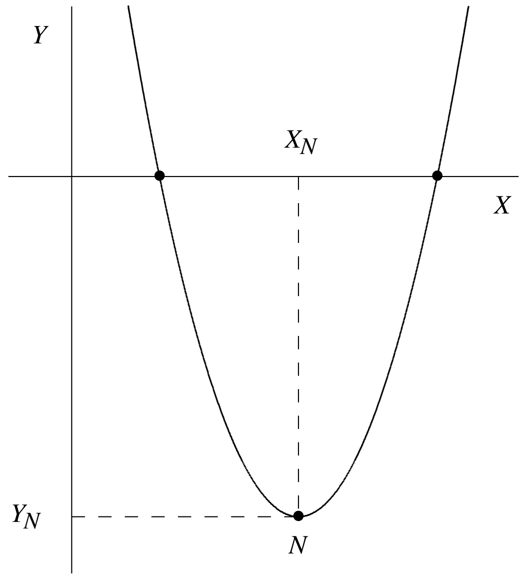

Figure 1: Quadratic showing the roots in relation to $N(x_N, y_N)$.

The quadratic formula's link with algebraic geometry is probably best visualised by expressing it in relation to the point $N(x_N, y_N)$ (see Figure 1), where $x_N = -b/2a$, and y_N is seen to be the geometric discriminant Δ_2 of the quadratic, with $\partial_2 = -4a\Delta_2$ [3]. If $y_N = 0$ then there is a double root at $x = x_N$; if $ay_N > 0$ then the roots are complex.

$$
\begin{aligned}
x_1, x_2 &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \\
&= x_N \pm \sqrt{\frac{-y_N}{a}}.
\end{aligned}
$$

4 The cubic

Substituting $n = 3$ in the general formula (2) generates the equivalent relationship for the cubic. In this case it yields the x-coordinates of the two turning points t_1, t_2. The cubic's maxima and minima are located symmetrically either side of the point of inflection $N(x_N, y_N)$, as shown in Figure 2.

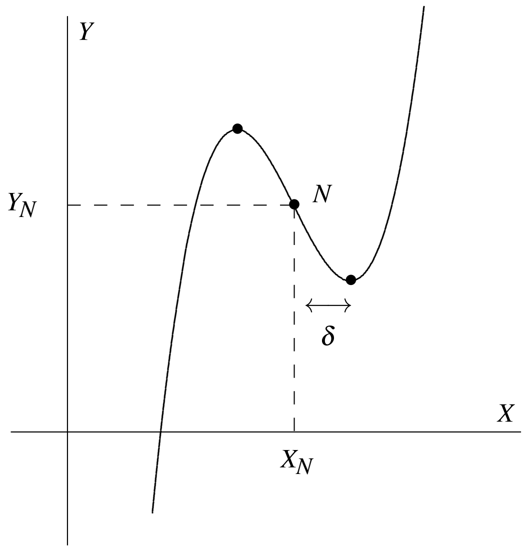

Figure 2: Cubic showing the turning points in relation to $N(x_N, y_N)$.

$$\begin{aligned} x_{t_1}, x_{t_2} &= \frac{-b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a}, \\ &= x_N \pm \delta. \end{aligned}$$

The quantity $b^2 - 3ac$ is therefore an algebraic discriminant *for the turning points of the cubic*. For example, if $b^2 = 3ac$ then $\delta = 0$ and the turning points coincide at $x = x_N$, and hence the slope at the point of inflection will be zero. Consequently if y_N is *also* zero, then the cubic has three equal roots at $x = x_N$. The product of the y-coordinates of the turning points is the geometric discriminant Δ_3 of the monic cubic [3].

5 The quartic

Substituting $n = 4$ in the general formula (2) yields the equivalent two-object relationship for the quartic, which generates the x-coordinates of the two points of inflection i_1, i_2 as follows.

$$x_{i_1}, x_{i_2} = \frac{-b}{4a} \pm \frac{\sqrt{b^2 - \frac{8}{3}ac}}{4a},$$
$$= x_N \pm \varepsilon.$$

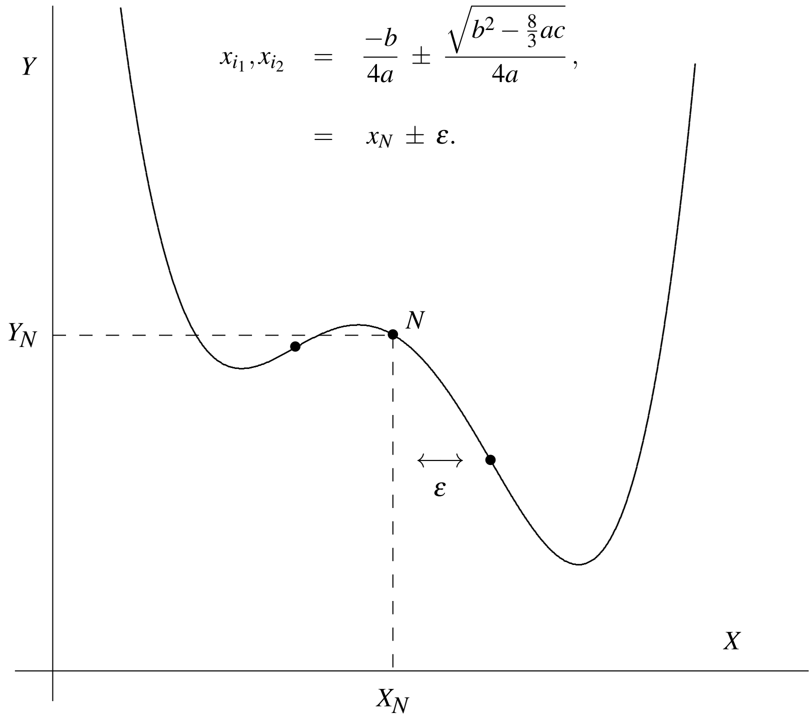

Figure 3: Quartic showing the two points of inflection in relation to $N(x_N, y_N)$.

Note that the two points of inflection are located symmetrically either side of $N(x_N, y_N)$ as shown in Figure 3. Thus $3b^2 - 8ac$ is an algebraic discriminant *for the points of inflection of the quartic*. For example, if $\varepsilon = 0$ then the points of inflection coincide at $x = x_N$, and there is only one turning point.

6 References

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3. Nickalls R. W. D. and Dye R. H. (1996). The geometry of the discriminant of a polynomial. *The Mathematical Gazette*; 80 (July), 279–285 (JSTOR). http://www.nickalls.org/dick/papers/maths/discriminant1996.pdf

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