# A proof of Nickalls' theorem on tangents and foci of a conic.

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R. W. D. Nickalls' interesting theorem [2] is a simple consequence of a more basic result, Theorem 1 below. I am not sure how well known this theorem is, but I recall it from my school days. I tracked it down in [3, p. 176], but the method of proof given here is different from that given by Salmon.

We use Nickalls' notation $\angle X\overset{\frown}{Y}Z$ to denote the *directed angle*, measured modulo 180°, from the line $YX$ to the line $YZ$; this angle is positive or negative according as the direction of rotation from $YX$ to $YZ$ is anticlockwise or clockwise. Also we introduce further notation, non-standard but useful. Choose an *initial line* (which will later be the $x$-axis when we come to introduce coordinates). The directed angle between this initial line and the line $PQ$ will be denoted by $(PQ)$. Then clearly

$$\angle X\overset{\frown}{Y}Z = (YZ) - (YX).$$

Note that all equalities between directed angles must be interpreted modulo 180°.

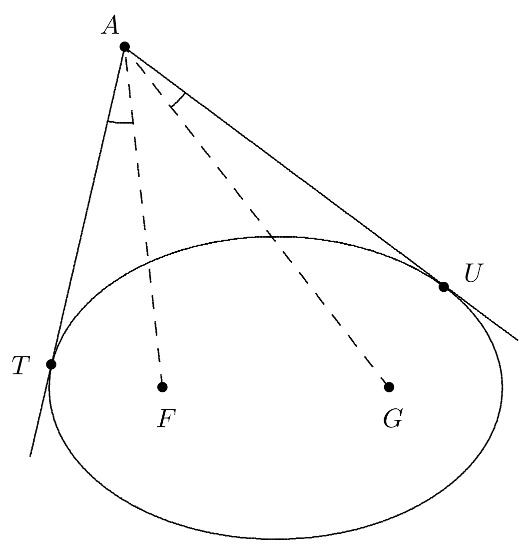


Figure 1

*Theorem 1.* Let $F$ and $G$ be the foci of a conic (if the conic is a parabola, then $G$ is the point at infinity on the axis), and let $AT$, $AU$ be the tangents to the conic from a point $A$ (if $A$ lies on the conic, these two tangents coincide). Then $\angle T\overset{\frown}{A}F = \angle G\overset{\frown}{A}U$. See Figure 1.

This result may also be written as $(AF)-(AT)=(AU)-(AG)$, or $(AT)+(AU)=(AF)+(AG)$.

*Proof.* We shall consider the proof for an ellipse. The proof for a hyperbola is similar, and the reader will easily supply a proof for a parabola. Let the equation of the ellipse be $x^2/a^2+y^2/b^2=1$, and let $(\lambda,\mu)$ be the coordinates of $A$. Now $(AT)+(AU)=(AF)+(AG)$ if, and only if,

$$\tan[(AT)+(AU)]=\tan[(AF)+(AG)].$$

Hence we need only prove that

$$\frac{m+n}{1-mn}=\frac{p+q}{1-pq}$$

where $m$, $n$, $p$, $q$ are the gradients of $AT$, $AU$, $AF$, $AG$. Now, as Nickalls remarks in [2], $m$ and $n$ are the roots of the quadratic in $M$

$$M^2(\lambda^2-a^2)-M(2\lambda\mu)+\mu^2-b^2=0$$

[1, p. 248]. Hence

$$\frac{m+n}{1-mn}=\frac{2\lambda\mu}{(\lambda^2-a^2)-(\mu^2-b^2)}.$$

Now the foci $F$ and $G$ have coordinates $(-ae,0)$ and $(ae,0)$, where $a^2e^2=a^2-b^2$. Hence $p=\mu/(\lambda+ae)$ and $q=\mu/(\lambda-ae)$, whence

$$\frac{p+q}{1-pq}=\frac{2\lambda\mu}{(\lambda^2-a^2)-(\mu^2-b^2)},$$

and the result follows immediately.

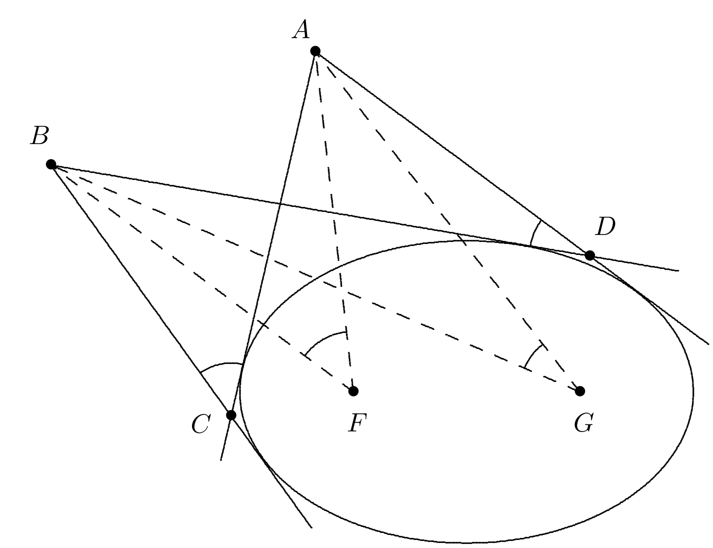


Figure 2

*Theorem 2* (Nickalls' theorem). Let $AC$ and $AD$ be the tangents from $A$ to a conic with foci $F$ and $G$, and let $BC$ and $BD$ be the tangents from $B$, as in Figure 2. Then

$$\angle A\overset{\frown}{C}B+\angle A\overset{\frown}{D}B=\angle A\overset{\frown}{F}B+\angle A\overset{\frown}{G}B\ .$$

*Proof.*

$$\begin{aligned}
\angle A\widehat{C}B + \angle A\widehat{D}B - \angle A\widehat{F}B - \angle A\widehat{G}B & \\
&= (CB) - (CA) + (DB) - (DA) - (FB) + (FA) - (GB) + (GA) \\
&= (CB) + (DB) - (FB) - (GB) - (CA) - (DA) + (FA) + (GA) \\
&= 0 - 0 = 0 \text{ by Theorem 1.}
\end{aligned}$$

Note that the theorem remains true if the other two points of intersection of the four tangents are used for $C$ and $D$.

Nickalls' Example 3 in [2] follows directly from Theorem 1. If we take $A$ to lie on the conic in Theorem 1, the two tangents from $A$ coincide, and we obtain the result that the lines joining $A$ to the foci are equally inclined to the tangent at $A$.

## References

1. S. L. Loney, *The elements of coordinate geometry. Part I*, Macmillan, London (1933).
2. R. W. D. Nickalls, A conic theorem generalised. *Math. Gaz.*, **84** (July 2000) pp. 232–241.
3. G. Salmon. *A treatise on conic sections* (5th edn.), Longmans, Green, Reader and Dyer, London (1869).