The quartic equation: alignment with an equivalent tetrahedron [1]

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1 Introduction

The lower polynomials are inextricably linked to the symmetries of polyhedra and Platonic solids [1,2,3], and the quartic is no exception; its *alter ego* is the regular tetrahedron [4]. In this article we present a solution to the problem of aligning the vertices of a tetrahedron with the roots of a particular quartic. After establishing the size of a quartic-equivalent tetrahedron, we derive a triple-angle expression for the alignment rotation, analogous to that for the cubic [5].

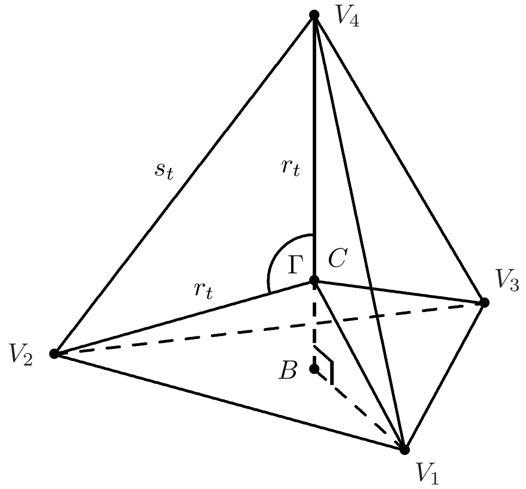

Figure 1:
A regular tetrahedron with centroid C, circumsphere radius r_t and edge length s_t. Γ is the angle subtended at C by an edge.

Since rotational symmetry considerations force us to consider only a regular tetrahedron, let such a tetrahedron have centroid C, edge length s_t and circumsphere radius r_t, as shown in Figure 1. Standard geometric properties we shall use are (a) $\cos\Gamma = -1/3$, where Γ is the angle subtended at C by an edge, (b) $s_t\sqrt{6} = 4r_t$, and (c) $CV_4 = 3CB$.

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Let $F(X) = aX^4 + bX^3 + cX^2 + dX + e$ be a quartic with real coefficients ($a \neq 0$) and let $P(x) = F(x + X_N)$ be its reduced form (Figure 2), where $X_N = -b/(4a)$. If $P(x)$ and $P'(x)$ intersect the Y-axis in points N_p and $N_{p'}$ respectively, then $P(x)$, with roots p_j $(j = 1, 2, 3, 4)$, can be expressed as [6]

$$P(x) = ax^4 - 6a\varepsilon^2 x^2 + y_{N_{p'}} x + y_{N_p} \tag{1}$$

where $\varepsilon^2 = (3b^2 - 8ac)/(48a^2)$, and $\pm\varepsilon$ are the x-coordinates of the two points of inflection. Let $G(x)$ be the reduced form of the resolvent sextic with roots $\pm g_i$ $(i = 1, 2, 3)$. Let the six tetrahedral mid-edge points (isomorphic with the roots of $G(x)$ [6]) be denoted as M_{ij} (Figure 2).

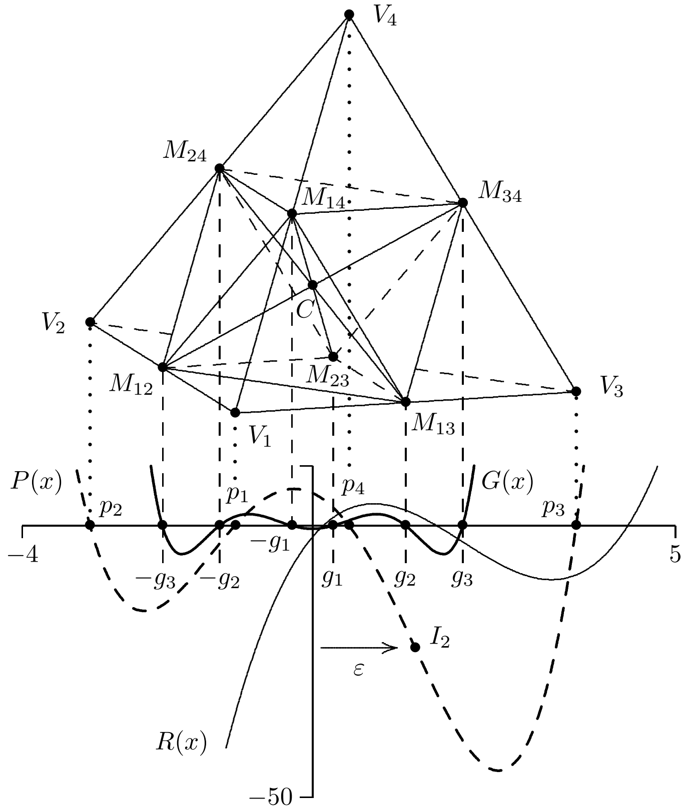

Figure 2:

Figure 2 shows a reduced quartic $P(x) = x^4 - 12x^2 - 6x + 6$ (dashed curve; $\varepsilon^2 = 2$) with roots p_j aligned with the four vertices V_j of an equivalent regular tetrahedron ($r_t = 3\varepsilon$). I_2 is a point of inflection. The roots $\pm g_i$ of the resolvent sextic $G(x)$ [6] are aligned with the mid-edge points M_{ij} which coincide with the vertices of a regular octahedron. Euler's resolvent cubic $R(x)$ [6] has three positive roots g_1^2, g_2^2, g_3^2. Note that the x-coordinate of a vertex V_j is the sum of the x-coordinates of the three adjacent M_{ij}, and hence the alignment reveals a 3D representation of Euler's solution [6]; for example, $p_3 = g_1 + g_2 + g_3$.

2 Size

By considering quartic configurations associated with an unambiguous tetrahedron orientation, it is easy to show (Figure 3) that the circumsphere radius r_t of a quartic-equivalent regular tetrahedron is 3ε.

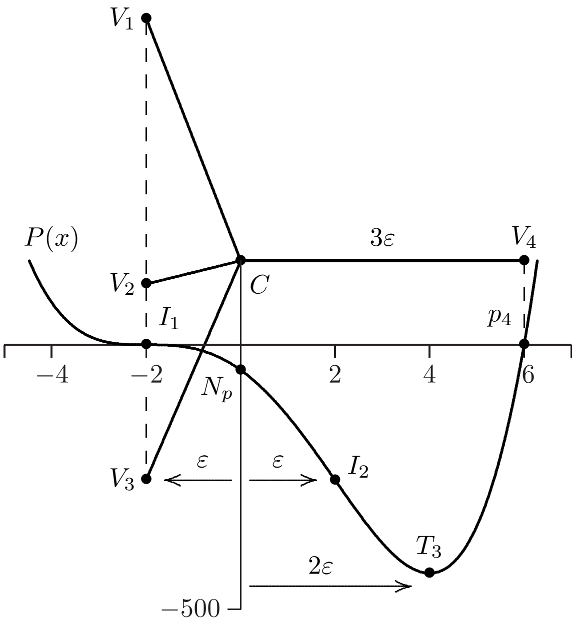

Figure 3:

Figure 3 shows a reduced quartic $P(x) = (x+2)^3(x-6)$ with $\varepsilon = 2$ and $y_{N_p} = -3a\varepsilon^4$, aligned with its 'equivalent' regular tetrahedron [3]. The four vertices V_j and centroid C of the tetrahedron are aligned with the quartic's roots p_j and N_p respectively. The x-coordinates of the points of inflection I_1, I_2 are $\pm\varepsilon$. Since $p_4 = 6 = 3\varepsilon$ and $CV_4 = 3CB$ (Figure 1), then $r_t = CV_4 = 3\varepsilon$.

[3]We can regard the four quartic roots of $P(x) = 0$ as being the x-coordinates of the ends of four root-vectors radiating symmetrically from the origin to the surface of a sphere with radius 3ε. For an equation of degree n we imagine n root-vectors, each of length $(n-1)\Omega$, radiating symmetrically in $\mathbb{R}^{n-1}$ (for the quartic $\Omega = \varepsilon$). When there are $n-1$ equal roots (as in Figure 3) then symmetry requirements constrain the remaining root-vector to be aligned with the x-axis; see Nickalls RWD (2011), A new bound for polynomials when all the roots are real, *The Mathematical Gazette*, 95, 520–526. http://www.nickalls.org/dick/papers/maths/bound2011.pdf

3 Orientation

Let the initial (default) configuration of the tetrahedron $(\theta, \phi = 0)$ be base down with the top vertex V_4 $(0, 0, 3\varepsilon)$ on the positive Z-axis, and with vertex V_1 $(0, -2\varepsilon\sqrt{2}, -\varepsilon)$ in the negative YZ half-plane (Figure 4). We will regard the rotation $+\theta$ as anticlockwise when viewed from the positive Y-axis (Figure 4), since this has the convenience that aligning V_4 with a positive quartic root is associated with a positive θ and vice versa. We will regard the rotation $+\phi$ as clockwise when viewed from V_4 (positive Z-axis).

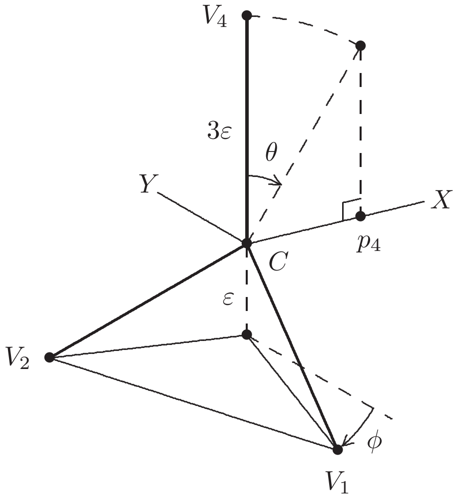

Figure 4:
Schematic of a regular tetrahedron with circumsphere radius $r_t = 3\varepsilon$ after an initial rotation ϕ about the Z-axis (CV_4) from the default position. The rotation θ about the Y-axis aligns the vertex V_4 with an arbitrary quartic root (p_4).

	X_0	Y_0	Z_0
V_1	0	$-2\varepsilon\sqrt{2}$	$-\varepsilon$
V_2	$-\varepsilon\sqrt{6}$	$+\varepsilon\sqrt{2}$	$-\varepsilon$
V_3	$+\varepsilon\sqrt{6}$	$+\varepsilon\sqrt{2}$	$-\varepsilon$
V_4	0	0	$+3\varepsilon$
$-g_2 \equiv M_{24}$	$-\varepsilon\sqrt{6}/2$	$+\varepsilon\sqrt{2}/2$	$+\varepsilon$
$g_3 \equiv M_{34}$	$+\varepsilon\sqrt{6}/2$	$+\varepsilon\sqrt{2}/2$	$+\varepsilon$
$-g_1 \equiv M_{14}$	0	$-\varepsilon\sqrt{2}$	$+\varepsilon$
$g_1 \equiv M_{23}$	0	$+\varepsilon\sqrt{2}$	$-\varepsilon$
$-g_3 \equiv M_{12}$	$-\varepsilon\sqrt{6}/2$	$-\varepsilon\sqrt{2}/2$	$-\varepsilon$
$g_2 \equiv M_{13}$	$+\varepsilon\sqrt{6}/2$	$-\varepsilon\sqrt{2}/2$	$-\varepsilon$

Table 1:
Coordinates of the tetrahedral vertices and mid-edge points associated with the initial position $(\theta, \phi = 0)$ as shown in Figure 4. The $\pm g_i$ are the x-coordinates of the mid-edge points M_{ij} (Figure 2).

3.1 Rotation matrix

Mathematically we perform the ϕ rotation about the Z-axis first, and then perform the θ rotation about the Y-axis, as then both rotations operate within the same axis frame. Since any subsequent rotation about the X-axis leaves the x-coordinates of the vertices unchanged, it suffices to consider only these two rotations.

The effect of this rotation sequence can be expressed as a combined rotation matrix R_{YZ} as follows, where we adopt the usual convention of writing the first action on the right and the last action on the left.

$$R_{YZ} = [R_Y(\theta)][R_Z(\phi)] = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If (X_0, Y_0, Z_0) are the initial coordinates of a point on the tetrahedron (see Table 1), then the final coordinates (X, Y, Z) after the sequence of two rotations are given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} X_0\cos\theta\cos\phi + Y_0\cos\theta\sin\phi + Z_0\sin\theta \\ -X_0\sin\phi + Y_0\cos\phi \\ -X_0\sin\theta\cos\phi - Y_0\sin\theta\sin\phi + Z_0\cos\theta \end{pmatrix}. \tag{2}$$

3.2 Euler's sign rule

The fundamental connection between tetrahedral geometry and the quartic equation is embodied in a sign rule of Euler [6], equivalent to

$$8g_1g_2g_3 = \frac{-y_{Np'}}{a}, \tag{3}$$

which defines the valid combinations of the $\pm g_i$ which generate the quartic roots, and hence links the rotations θ, ϕ to the quartic coefficients.

Substituting the initial coordinates (Table 1) of the mid-edge points associated with V_4 (and hence with θ), namely M_{14}, M_{24}, M_{34} (Figure 2), into the relationship (from (2))

$$X = X_0\cos\theta\cos\phi + Y_0\cos\theta\sin\phi + Z_0\sin\theta, \tag{4}$$

gives the following post-rotation values [4]:

$$\begin{cases} -g_1 \equiv X_{M_{14}} = -\varepsilon\sqrt{2}\cos\theta\sin\phi + \varepsilon\sin\theta, \\ -g_2 \equiv X_{M_{24}} = \dfrac{-\varepsilon\sqrt{6}}{2}\cos\theta\cos\phi + \dfrac{\varepsilon\sqrt{2}}{2}\cos\theta\sin\phi + \varepsilon\sin\theta, \\ g_3 \equiv X_{M_{34}} = \dfrac{+\varepsilon\sqrt{6}}{2}\cos\theta\cos\phi + \dfrac{\varepsilon\sqrt{2}}{2}\cos\theta\sin\phi + \varepsilon\sin\theta. \end{cases} \tag{5}$$

[4] see Appendix 6.1

Substituting these in (3) with $\phi_i = \phi + 2\pi(i-1)/3$, $(i = 1, 2, 3)$ gives

$$8(-g_1)(-g_2)g_3 = 8\prod_{i=1}^{3}\left(\varepsilon\sin\theta - \varepsilon\sqrt{2}\cos\theta\sin\phi_i\right) = \frac{-y_{N_{p'}}}{a},$$

which, after expanding and regrouping, gives

$$\sin^3\theta - U_1\sqrt{2}\sin^2\theta\cos\theta + 2U_2\sin\theta\cos^2\theta - 2U_3\sqrt{2}\cos^3\theta = \frac{-y_{N_{p'}}}{8a\varepsilon^3}, \qquad (6)$$

where

$$\begin{cases} U_1 = \sin\phi_1 + \sin\phi_2 + \sin\phi_3 = 0, \\ U_2 = \sin\phi_1\sin\phi_2 + \sin\phi_1\sin\phi_3 + \sin\phi_2\sin\phi_3 = \dfrac{-3}{4}, \\ U_3 = \sin\phi_1\sin\phi_2\sin\phi_3 = \dfrac{-\sin 3\phi}{4}. \end{cases}$$

After substituting for U_1, U_2, U_3 (6) reduces to

$$2\sin^3\theta - 3\sin\theta\cos^2\theta + \sqrt{2}\sin 3\phi\cos^3\theta = \frac{-y_{N_{p'}}}{4a\varepsilon^3},$$

and since $\cos^2\theta = 1 - \sin^2\theta$ we have, finally,

$$\sin 3\phi = \frac{\{-y_{N_{p'}}/(4a\varepsilon^3)\} + 3\sin\theta - 5\sin^3\theta}{\sqrt{2}\cos^3\theta}, \qquad (7)$$

which yields the rotation ϕ which aligns the remaining three vertices and quartic roots.

In practice, if an initial rotation θ aligns one of the vertices with an arbitrary root (say, $p = 3\varepsilon\sin\theta$), the full alignment associated with ϕ is readily confirmed (see Example), since using (4) in conjunction with Table 1, and noting that $2\sqrt{2} = 3\sin\Gamma$ (Figure 1), it can be shown that the x-coordinates of the remaining three vertices are given by [5]:

$$X_{V_i} = -\varepsilon\sin\theta - 2\varepsilon\sqrt{2}\sin\phi_i\cos\theta\,, \qquad (i = 1, 2, 3) \qquad (8)$$

$$= \frac{-p}{3} - \sin\Gamma\sin\phi_i\sqrt{(3\varepsilon)^2 - p^2}. \qquad (9)$$

Remark. Since $p = 3\varepsilon\sin\theta$ is an arbitrary root of (1) we can use the equation $P(3\varepsilon\sin\theta) = 0$ to express $y_{N_{p'}}$ in terms of y_{N_p} and hence derive from (7) the useful complementary form

$$\sin 3\phi = \frac{\{y_{N_p}/(3a\varepsilon^4)\} - 6\sin^2\theta + 7\sin^4\theta}{4\sqrt{2}\sin\theta\cos^3\theta}. \qquad (10)$$

Note that (10) can be regarded as representing the quartic analogue of the cubic three-real-root form $\sin 3\psi = y_N/h$ [5].

[5] see Appendix 6.2

4 Example

Determine the circumsphere radius r_t of a regular tetrahedron equivalent to the quartic $P(x) = x^4 - 15x^2 - 10x + 24 = 0$. Align V_4 with root $p = 1$ and determine the rotation ϕ which aligns the remaining vertices and roots.

Comparison with (1) yields $a = 1$, $y_{Np'} = -10$ and $\varepsilon^2 = 5/2$, and so for the circumsphere radius we have $r_t = 3\varepsilon = 3\sqrt{5/2}$.

Aligning V_4 with the root $p = 1$, we have $X_{V_4} = 3\varepsilon \sin\theta = 1$, giving $\theta \approx 12{\cdot}17°$. Substituting for $a, \varepsilon, \theta, y_{Np'}$ into (7) then gives $\sin 3\phi \approx 0{\cdot}922$, and hence $\phi \approx 22{\cdot}4°$.

Finally as a check, we can substitute for $p, \varepsilon, \theta, \phi$ into (8) or (9) to confirm that the x-coordinates of the remaining vertices ($X_{V_1} = -2$, $X_{V_2} = -3$, $X_{V_3} = 4$) agree with the roots, which they do.

5 References

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6 Appendix

6.1 Appendix 1 (see §3.2 and equation 5)

Note that by letting $\phi_i = \phi + 2\pi(i-1)/3$, $(i = 1, 2, 3)$, the x-coordinates of the three tetrahedral mid-edge points M_{i4} associated with the vertex V_4 can be expressed more conveniently as

$$X_{M_{i4}} = \varepsilon \sin\theta - \varepsilon\sqrt{2}\cos\theta \sin\phi_i.$$

M_{14} ($i = 1$)

Since $\phi_1 = \phi + 0$, it follows from (5) that $X_{M_{14}} = \varepsilon \sin\theta - \varepsilon\sqrt{2}\cos\theta\sin\phi_1$.

M_{24} ($i = 2$)

Since $\phi_2 = \phi + 120^\circ$ we can substitute $\phi = \phi_2 - 120^\circ$ in $X_{M_{24}}$, giving

$$\begin{aligned} X_{M_{24}} &= \varepsilon\sin\theta - \frac{\varepsilon\sqrt{6}}{2}\cos\theta\cos(\phi_2 - 120^\circ) + \frac{\varepsilon\sqrt{2}}{2}\cos\theta\sin(\phi_2 - 120^\circ), \\ &= \varepsilon\sin\theta - \frac{\varepsilon\sqrt{6}}{2}\cos\theta\{\cos\phi_2\cos 120^\circ + \sin\phi_2\sin 120^\circ\} \\ &\quad + \frac{\varepsilon\sqrt{2}}{2}\cos\theta\{\sin\phi_2\cos 120^\circ - \cos\phi_2\sin 120^\circ\}, \end{aligned}$$

which expands to give

$$\begin{aligned} X_{M_{24}} = \varepsilon\sin\theta - \cos\theta&\left\{\cos\phi_2\left(\frac{\varepsilon\sqrt{6}}{2}\right)\left(\frac{-1}{2}\right) + \sin\phi_2\left(\frac{\varepsilon\sqrt{6}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\right\} \\ + \cos\theta&\left\{\sin\phi_2\left(\frac{\varepsilon\sqrt{2}}{2}\right)\left(\frac{-1}{2}\right) - \cos\phi_2\left(\frac{\varepsilon\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\right\}, \end{aligned}$$

which reduces to

$$\begin{aligned} X_{M_{24}} &= \varepsilon\sin\theta + \cos\theta\sin\phi_2\left\{\frac{-\varepsilon\sqrt{18}}{4} - \frac{\varepsilon\sqrt{2}}{4}\right\}, \\ &= \varepsilon\sin\theta - \varepsilon\sqrt{2}\cos\theta\sin\phi_2. \end{aligned}$$

M_{34} ($i = 3$)

Similarly, substituting $\phi = \phi_3 - 240^\circ$ in $X_{M_{34}}$, then gives

$$X_{M_{34}} = \varepsilon\sin\theta - \varepsilon\sqrt{2}\cos\theta\sin\phi_3.$$

6.2 Appendix 2 (see equation 8)

Since $2\sqrt{2} = 3\sin\Gamma$ (because $\cos\Gamma = -1/3$, see Figure 1), and $p = 3\varepsilon\sin\theta$, equation 8 can be expressed as

$$X_{V_i} = \frac{-p}{3} - \sin\Gamma\sin\phi_i(3\varepsilon\cos\theta),$$

and since $\cos\theta = \sqrt{1 - \sin^2\theta}$ we can write

$$\begin{aligned} X_{V_i} &= \frac{-p}{3} - \sin\Gamma\sin\phi_i\sqrt{9\varepsilon^2 - 9\varepsilon^2\sin^2\theta}, \\ &= \frac{-p}{3} - \sin\Gamma\sin\phi_i\sqrt{(3\varepsilon)^2 - p^2}. \end{aligned}$$

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