The rotating Pulfrich effect — derivation of equations

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Chapter 3

The rotating Pulfrich effect: derivation of equations[1]

Here we detail the derivation of the equations presented in the Appendix to the paper:-

- Nickalls RWD (1986). The rotating Pulfrich effect, and a new method of determining visual latency differences. *Vision Research*; 26, 367–372. (http://www.nickalls.org/dick/papers/pulfrich/pulfrich1986.pdf)

3.1 The general case

Consider that the eyes (L, R; separation $2a$) view an object P (the *target*) rotating clockwise about the center O with constant angular velocity ω. Let the eyes (L,R) be a distance d from the center of rotation O such that the line LR is parallel to the y-axis (see Figure 3.1).

If the target P is associated with angle θ, then let P' lag behind P by angle ϕ (due to a filter F in front of the right eye). The locus I of the apparent position is given by the intersection of the lines RP and LP'.

Let the lines RP' and LP be given by

$$RP' \qquad y = m_1(x+d)-a, \tag{3.1}$$

$$LP \qquad y = m_2(x+d)+a, \tag{3.2}$$

where

$$m_1 = \frac{r\sin\theta + a}{r\cos\theta + d}, \tag{3.3}$$

$$m_2 = \frac{r\sin(\theta-\phi) - a}{r\cos(\theta-\phi) + d}. \tag{3.4}$$

Solving these for x and y gives

$$x = \left(\frac{2a}{m_1 - m_2}\right) - d, \tag{3.5}$$

$$y = a\left(\frac{m_1+m_2}{m_1-m_2}\right). \tag{3.6}$$

[1] http://www.nickalls.org/dick/papers/pulfrich/pg-rotating.pdf

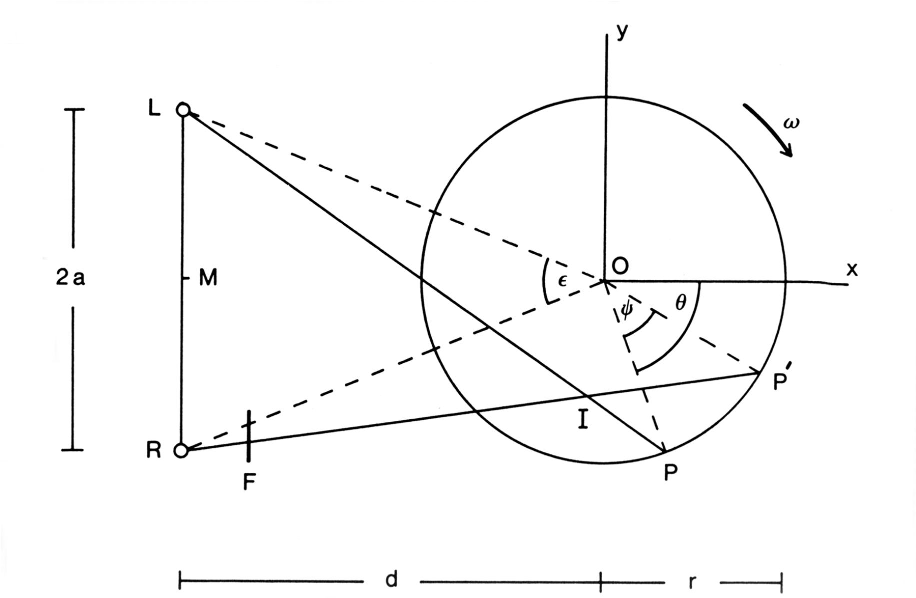

Figure 3.1:
Diagram (from above) showing the relative positions in the general case (see text).

3.1.1 Determine $\mathbf{m_1 - m_2}$

$$\begin{aligned} m_1 - m_2 &= \left(\frac{r\sin\theta + a}{r\cos\theta + d}\right) - \left(\frac{r\sin(\theta-\phi) - a}{r\cos(\theta-\phi) + d}\right) \\ &= \frac{(r\sin\theta + a)\{r\cos(\theta-\phi)+d\} - (r\cos\theta + d)\{r\sin(\theta-\phi) - a\}}{(r\cos\theta + d)\{r\cos(\theta-\phi)+d\}}. \end{aligned}$$

Expanding and regrouping gives

$$m_1 - m_2 = \frac{\left\{\begin{array}{l} r^2\{\sin\theta\cos(\theta-\phi) - \cos\theta\sin(\theta-\phi)\} + rd\{\sin\theta - \sin(\theta-\phi)\} \\ \qquad\qquad + ar\{\cos\theta + \cos(\theta-\phi)\} + 2ad \end{array}\right\}}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2}.$$

Since

$$\sin\theta\cos(\theta-\phi) - \cos\theta\sin(\theta-\phi) \equiv \sin\{\theta - (\theta-\phi)\} \equiv \sin\phi$$

this simplifies to

$$m_1 - m_2 = \frac{r^2\sin\phi + rd\{2\cos(\theta-\phi/2)\sin(\phi/2)\} + ar\{2\cos(\theta-\phi/2)\cos(\phi/2)\} + 2ad}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2},$$

i.e.,

$$m_1 - m_2 = \frac{r^2\sin\phi + 2rd\cos(\theta-\phi/2)\sin(\phi/2) + 2ar\cos(\theta-\phi/2)\cos(\phi/2) + 2ad}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2}. \tag{3.7}$$

3.1.2 Determine $\mathbf{m_1 + m_2}$

$$\begin{aligned} m_1 + m_2 &= \left(\frac{r\sin\theta + a}{r\cos\theta + d}\right) + \left(\frac{r\sin(\theta-\phi) - a}{r\cos(\theta-\phi) + d}\right) \\ &= \frac{(r\sin\theta + a)\{r\cos(\theta-\phi)+d\} + (r\cos\theta + d)\{r\sin(\theta-\phi) - a\}}{(r\cos\theta + d)\{r\cos(\theta-\phi)+d\}}. \end{aligned}$$

Expanding and regrouping gives

$$m_1 + m_2 = \frac{\left\{\begin{array}{r} r^2\{\sin\theta\cos(\theta-\phi) + \cos\theta\sin(\theta-\phi)\} + rd\{\sin\theta + \sin(\theta-\phi)\} \\ + ar\{\cos(\theta-\phi) - \cos\theta\} \end{array}\right\}}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2}.$$

Since

$$\sin\theta\cos(\theta-\phi) + \cos\theta\sin(\theta-\phi) = \sin\{\theta + (\theta-\phi)\} = \sin(2\theta-\phi),$$

this simplifies to

$$m_1 + m_2 = \frac{\left\{\begin{array}{r} r^2\sin(2\theta-\phi) + rd\{2\sin(\theta-\phi/2)\cos(\phi/2)\} \\ + ar\{-2\sin(\theta-\phi/2)\sin(-\phi/2)\} \end{array}\right\}}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2}.$$

Since $\sin(-\phi/2) = -\sin(\phi/2)$ then

$$-2\sin(\theta-\phi/2)\sin(-\phi/2)\} \equiv \{2\sin(\theta-\phi/2)\sin(\phi/2),$$

and so we have

$$m_1 + m_2 = \frac{r^2\sin(2\theta-\phi) + rd\{2\sin(\theta-\phi/2)\cos(\phi/2)\} + ar\{2\sin(\theta-\phi/2)\sin(\phi/2)\}}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2}.$$

Since $\sin(2\theta-\phi) \equiv 2\sin(\theta-\phi/2)\cos(\theta-\phi/2)$ we can write

$$m_1 + m_2 = \frac{\left\{\begin{array}{r} 2r^2\sin(\theta-\phi/2)\cos(\theta-\phi/2) + 2rd\{\sin(\theta-\phi/2)\cos(\phi/2)\} \\ + 2ar\{\sin(\theta-\phi/2)\sin(\phi/2)\} \end{array}\right\}}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2},$$

i.e.,

$$m_1 + m_2 = \frac{2r\sin(\theta-\phi/2)\{r\cos(\theta-\phi/2) + d\cos(\phi/2) + a\sin(\phi/2)\}}{r^2\cos\theta\cos(\theta-\phi) + rd\{\cos\theta + \cos(\theta-\phi)\} + d^2}. \qquad (3.8)$$

3.1.3 Solve for x

From above we have

$$x = \left(\frac{2a}{m_1 - m_2} - d\right) = \frac{2a - d(m_1 - m_2)}{m_1 - m_2}.$$

Substituting for $(m_1 - m_2)$, and after some manipulation, we eventually get

$$x = \frac{2ar^2\cos\theta\cos(\theta-\phi) - r^2 d\sin\phi + 2rd\cos(\theta-\phi/2)\{a\cos(\phi/2) - d\sin(\phi/2)\}}{2ad + r^2\sin\phi + 2r\cos(\theta-\phi/2)\{a\cos(\phi/2) + d\sin(\phi/2)\}}. \tag{3.9}$$

Now we develop $2ar^2\cos\theta\cos(\theta-\phi)$ by expressing it in terms of $\cos(\theta-\phi/2)$, using the following identities:

$$\begin{cases} \cos\theta\cos(\theta-\phi) \equiv \dfrac{1}{2}\{\cos\phi + \cos(2\theta-\phi)\}, \\ \cos(2\theta-\phi) \equiv 2\cos^2(\theta-\phi/2) - 1. \end{cases} \tag{3.10}$$

Combining the two identities 3.10 then gives

$$2ar^2\cos\theta\cos(\theta-\phi) \equiv 2ar^2\cos^2(\theta-\phi/2) - ar^2 + ar^2\cos\phi,$$

which now allows us to substitute for $\cos(\theta-\phi)$ in equation 3.9 and hence obtain x as a function of $\cos(\theta-\phi/2)$ as follows

$$x = \frac{\left\{\begin{array}{l} 2ar^2\cos^2(\theta-\phi/2) - ar^2 + r^2(a\cos\phi - d\sin\phi) \\ \qquad\qquad + 2rd\cos(\theta-\phi/2)\{a\cos(\phi/2) - d\sin(\phi/2)\} \end{array}\right\}}{2ad + r^2\sin\phi + 2r\cos(\theta-\phi/2)\{a\cos(\phi/2) + d\sin(\phi/2)\}}. \tag{3.11}$$

3.1.4 Solve for y

From above we have

$$y = a\left(\frac{m_1+m_2}{m_1-m_2}\right).$$

Substituting for $(m_1 - m_2)$ and $(m_1 + m_2)$ we get

$$y = a\left(\frac{2r\sin(\theta-\phi/2)\{r\cos(\theta-\phi/2) + d\cos(\phi/2) + a\sin(\phi/2)\}}{r^2\sin\phi + 2rd\cos(\theta-\phi/2)\sin(\phi/2) + 2ar\cos(\theta-\phi/2)\cos(\phi/2) + 2ad}\right),$$

which after regrouping of terms gives

$$y = \frac{2ar\sin(\theta-\phi/2)\{r\cos(\theta-\phi/2) + d\cos(\phi/2) + a\sin(\phi/2)\}}{2ad + r^2\sin\phi + 2r\cos(\theta-\phi/2)\{a\cos(\phi/2) + d\sin(\phi/2)\}}.$$

Finally, we expand $r^2\sin\phi \to 2r^2\sin(\phi/2)\cos(\phi/2)$ and then divide throughout by 2, to give

$$y = \frac{ar\sin(\theta-\phi/2)\{r\cos(\theta-\phi/2) + d\cos(\phi/2) + a\sin(\phi/2)\}}{ad + r^2\sin(\phi/2)\cos(\phi/2) + r\cos(\theta-\phi/2)\{a\cos(\phi/2) + d\sin(\phi/2)\}}. \tag{3.12}$$

Equations 3.11 & 3.12 are therefore the parametric equations of the path I, as θ increases from 0 to 360 deg. When P rotates clockwise θ is considered to be +ve; with anti-clockwise rotation θ is considered to be −ve.

Now equation 3.11 is quadratic in $\cos(\theta-\phi/2)$ and so θ can be eliminated to give the Cartesian equation, by substituting the roots of equation 3.11 for $\cos(\theta-\phi/2)$ into equation 3.12. In the general case this gives rise to a complicated 'higher' curve which is symmetric about the x-axis, as shown in figure 3.2 (from Nickalls, 1986). The two curves shown in figure 3.2 represent equations 3.11 and 3.12 for a range of values of the parameter ϕ/ε, with P rotating clockwise. When $\phi/\varepsilon < 1$ then the Pulfrich construction (P_x) is also clockwise. When $\phi/\varepsilon > 1$ then the locus is anti-clockwise. The special 'transition' case, when $\phi/\varepsilon = 1$, is discussed in the next section.

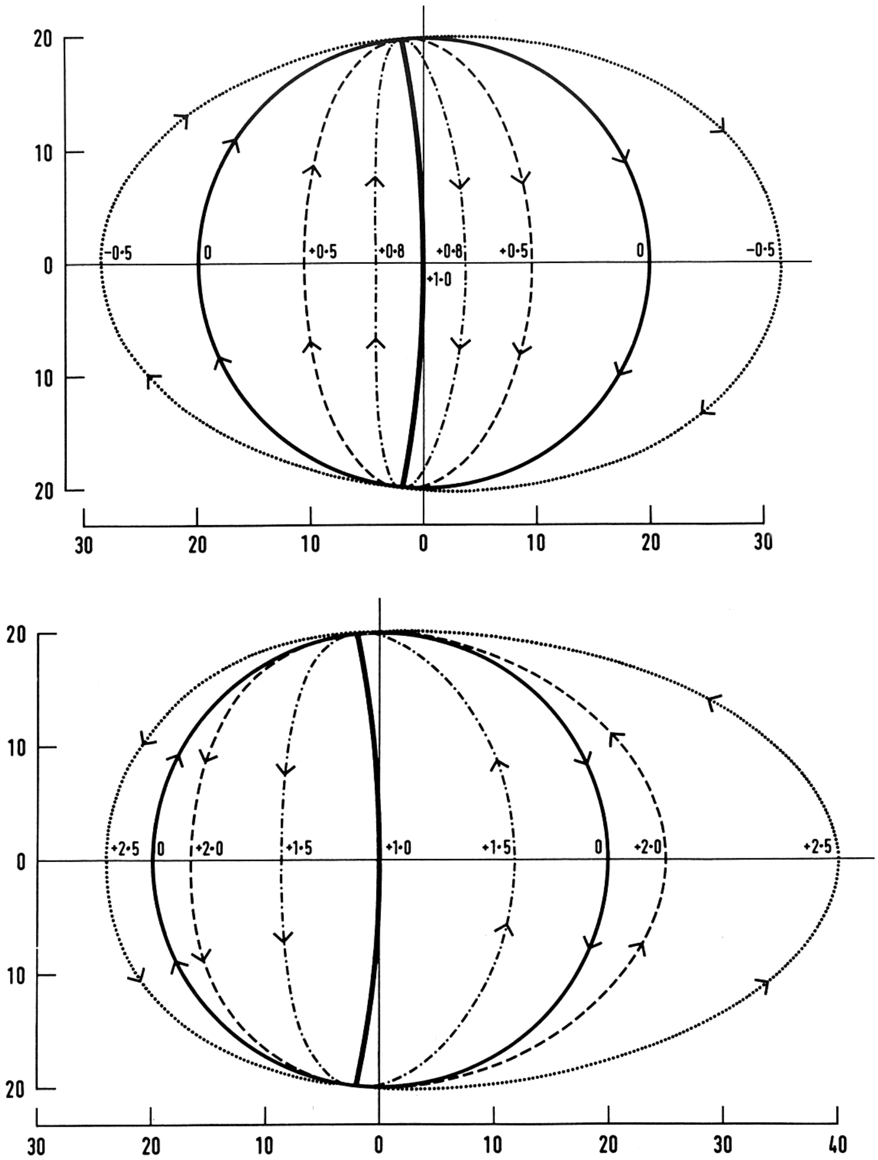

Figure 3.2:

These graphs show the theoretical curves (Equations 3.11 and 3.12) indicating the predicted apparent paths associated with different latency differences (i.e., with different filters) while fixating a vertical rod, rotating clockwise in a horizontal circle, from a distance of 200 cm to the left of the centre, as in the arrangement shown in Fig. 3.1 ($2a = 6{\cdot}6$ cm; $\omega = 45{\cdot}1$ rpm; rod is 20 cm from the centre; axes are in cm.) The parameter (latency ratio) is the latency difference expressed as a multiple of that required to see "transition" under these circumstances (7 msec). The +ve sign indicates that the filter is in front of the right eye; −ve sign indicates the filter is in front of the left eye. Arrows indicate the direction of apparent rotation. Thick circle (latency ratio = 0) indicates the actual path of the rotating rod ($\Delta t = 0$). The thick arc (latency ratio = +1) indicates the apparent path at "transition" ($\Delta t = 7$ msec).

Top: Latency ratios from $-0{\cdot}5$ to $+1$, clockwise rotation.

Bottom: Latency ratios from $+1$ to $+2{\cdot}5$, anticlockwise rotation.

3.2 The 'transition' condition ($\phi/\varepsilon = 1$)

This is the special 'transition' case which is associated with the condition $\phi = \varepsilon$. In this case the locus I is an arc of the circle LRO as shown in the following figure.

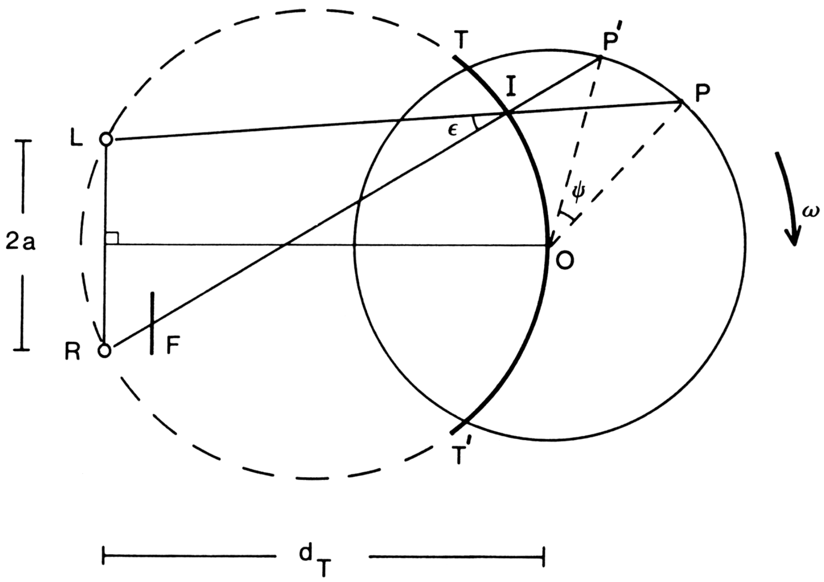

Figure 3.3:
Diagram (from above) showing the relative positions in the general case (see text).

At transition I passes through the center O and so

$$\tan(\phi/2) = a/d \tag{3.13}$$

Substituting equation 3.13 into equation 3.11 gives

$$x = \frac{-(r^2/d)(a^2+d^2)\sin^2(\theta-\phi/2)}{a^2+d^2+r^2+2r\sqrt{a^2+d^2}\cos(\theta-\phi/2)}, \tag{3.14}$$

and

$$y = \pm\sin(\theta-\phi/2)\left\{\frac{(r/d)(a^2+d^2)\{r\cos(\theta-\phi/2)+\sqrt{a^2+d^2}}{a^2+d^2+r^2+2r\sqrt{a^2+d^2}\cos(\theta-\phi/2)}\right\}. \tag{3.15}$$

Eliminating θ from equations 3.14 and 3.15 gives

$$y^2 = -x\left(\frac{a^2+d^2}{d}+x\right), \tag{3.16}$$

which represents a circle passing through L,R and the origin O, where $R = (a^2 + d^2)/(2d)$ as shown in the figure above. The equation can therefore be expressed in terms of R as follows.

$$y^2 = -x(2R+x).$$

It is important to note that the path at 'transition' is not necessarily the whole circle as, depending on the conditions, only part of the arc is mathematically 'real'. The limits are found by solving equation 3.15 for θ as follows.

Conveniently equation 3.15 can be turned into a quadratic by using the identity $\sin^2(\theta-\phi/2) \equiv 1-\cos^2(\theta-\phi/2)$ which generates

$$\left\{\begin{array}{r}(r^2/d)(a^2+d^2)\cos^2(\theta-\phi/2)-2xr\sqrt{a^2+d^2}\cos(\theta-\phi/2)\qquad\qquad\\ -x(a^2+d^2+r^2)-(r^2/d)(a^2+d^2)\end{array}\right\}=0.$$

Dividing throughout by the coefficient of $\cos^2(\theta-\phi/2)$ and rearranging we get

$$\cos^2(\theta-\phi/2)-\left(\frac{2xd}{r\sqrt{a^2+d^2}}\right)\cos(\theta-\phi/2)=\frac{xd}{r^2}+\frac{xd}{a^2+d^2}+1.$$

We can now use the method of 'completing the square' to solve the quadratic by adding the term $(xd/(r\sqrt{(}a^2+d^2))^2$ to both sides as follows.

$$\left(\cos(\theta-\phi/2)-\frac{xd}{r\sqrt{a^2+d^2}}\right)^2=\left(\frac{xd}{r\sqrt{a^2+d^2}}\right)^2+\frac{xd}{r^2}+\frac{xd}{a^2+d^2}+1,$$

which gives

$$\left(\cos(\theta-\phi/2)-\frac{xd}{r\sqrt{a^2+d^2}}\right)^2=\left(\frac{xd}{r^2}+1\right)\left(\frac{xd}{a^2+d^2}+1\right),$$

and so we have

$$\cos(\theta-\phi/2)=\frac{xd}{r\sqrt{a^2+d^2}}\pm\sqrt{\left(\frac{xd}{r^2}+1\right)\left(\frac{xd}{a^2+d^2}+1\right)}. \qquad (3.17)$$

Now since from the geometry we have $R^2=a^2+(d-R)^2$ then

$$2R=\frac{a^2+d^2}{d},$$

and so finally we have

$$\cos(\theta-\phi/2)=x\sqrt{\frac{d}{2Rr^2}}\pm\sqrt{\left(\frac{xd}{r^2}+1\right)\left(\frac{x}{2R}+1\right)}. \qquad (3.18)$$

It follows therefore that for equation 3.15 to yield 'real' solutions we need the contents of each of the square-root terms to be ≥ 0, that is we need $d/R \geq 0$ and

$$\left(\frac{xd}{r^2}+1\right)\left(\frac{x}{2R}+1\right)\geq 0,$$

for which we need either

$$\left(\frac{xd}{r^2}+1\right)\geq 0 \quad \text{AND} \quad \left(\frac{x}{2R}+1\right)\geq 0$$

or

$$\left(\frac{xd}{r^2}+1\right)\leq 0 \quad \text{AND} \quad \left(\frac{x}{2R}+1\right)\leq 0,$$

where d, r, and R are of course all negative owing to our chosen coordinate system (centered at the center of rotation) and configuration (viewing from the left).

However, x cannot be less than $-2R$ (since the locus is the arc of the circle radius R), and so only the first condition is relevant for a real locus, i.e

$$x \geq \frac{-r^2}{d} \qquad \text{AND} \qquad x \geq -2R,$$

which is consistent with the single condition

$$x \geq \frac{-r^2}{d}. \tag{3.19}$$

3.2.1 Viewing configurations

There are three configurations which may now be considered, depending on whether the points L, R are *outside* the circle, *on* the circle, or *inside* the circle. In practice however, since the observer is outside the circle of rotation, we need only consider the case for which L, R lie outside the circle ($a^2 + d^2 > r^2$), i.e.,

$$\frac{-r^2}{d} > -\left(\frac{a^2 + d^2}{d}\right),$$

for which the condition $x \geq -r^2/d$ is sufficient, as this is always consistent with $x \geq -(a^2 + d^2)/d$, and hence we have

$$\frac{-r^2}{d} \leq x \leq 0. \tag{3.20}$$

3.3 Related papers

The following related papers address the visual and/or geometric aspects of the rotating Pulfrich effect. They are all available from http://www.nickalls.org/

- Nakamizo S, Nawae H, and Nickalls RWD (1998). Precision of the rotating Pulfrich technique for determining visual latency difference is significantly greater if viewing distance is varied, than if angular velocity is varied. *Perception*; 27 (Supplement), 97. (Abstracts of the 21st European Conference on Visual Perception; Oxford, England; 24-28 August 1998). http://www.perceptionweb.com/abstract.cgi?id=v980346

- Nakamizo S, Nickalls RWD and Nawae H (2004). Visual latency difference determined by two 'rotating' Pulfrich techniques. *Swiss Journal of Psychology*, 63, 201–205.
http://www.nickalls.org/dick/papers/pulfrich/pulfrich2004.pdf

- Nickalls RWD (1986). A line-and-conic theorem having a visual correlate. *Mathematical Gazette*; 70 (March), 27–29, (JSTOR). http://www.nickalls.org/dick/papers/maths/lineandconic1986.pdf

- Nickalls RWD (1996). The influence of target angular velocity on visual latency difference determined using the rotating Pulfrich effect. *Vision Research*; 36, 2865–2872.
http://www.nickalls.org/dick/papers/pulfrich/pulfrich1996.pdf

- Nickalls RWD (2000). A conic theorem generalised: directed angles and applications. *Mathematical Gazette*; 84 (July), 232–241, (JSTOR). http://www.nickalls.org/dick/papers/maths/conicthm2000.pdf

- Nickalls RWD, Kazachkov AR, Vasylevska Yu.V and Kalinin VV (2002). Motional visual illusions on-line. *Proceedings of the 2002 International Conference on Information and Communication Technologies in Education* (ICTE2002); Badajoz Spain; November 13–16, 2002; p 1320–1324 (ISBN: Coleccion-84-95251-76-0).
http://www.nickalls.org/dick/papers/pulfrich/spain2002pulfrich.pdf